Cayley–Menger coordinates

(distance geometry/embedding of distances/Cayley–Menger determinants/incomplete embedding problem/nuclear magnetic resonance)

Manfred J. Sippl†‡ and Harold A. Scheraga†§
†Baker Laboratory of Chemistry, Cornell University, Ithaca, NY 14853-1301; and ‡Institute for General Biology, Biochemistry and Biophysics, University of Salzburg, Erzabt Klotz Strasse 11, A-5020 Salzburg, Austria

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ABSTRACT A major obstacle in applying distance geometry techniques is the analytical complexity of the Cayley–Menger determinants that are used to characterize euclidian spaces in terms of distances between points. In this paper we show that, with the aid of a theorem of Jacobi, the complex Cayley–Menger determinants can be replaced by simpler determinants, and we derive the concept of Cayley–Menger coordinates, a coordinate system in terms of which each point of $E^n$ is characterized by $n + 1$ distances to $n + 1$ points of a reference. We also show that this coordinate system provides a natural norm for the incomplete embedding problem. This paper provides the tools to treat the problem of finding an incomplete distance matrix so that our previous procedure can then be used to embed the corresponding structure in a three-dimensional space.

Our interest in distance geometry techniques stems from a problem in nuclear magnetic resonance (NMR) studies of macromolecules (1). NMR techniques yield a limited number of distances between a limited number of points. Together with the known chemical structure of the molecule (e.g., bond lengths and bond angles), the observed distances yield an incomplete distance matrix $D^*$. The problem is to find a complete matrix $D$—i.e., to fill $D^*$—so that $D$ is embeddable in $E^3$. Formally, we have the special incomplete embedding problem.

Given a partially filled matrix $D^*$ of interpoint distances $d_{ij}$, $i = 1, \ldots, m$; $(i, j) \in I$, which are known to be derived from a system of points in $E^n$, fill the matrix $D^*$ so that the complete matrix $D$ is embeddable in $E^n$. $I$ is the set of all pairs $(i, j)$ for which $d_{ij}$ is specified in $D^*$.

Our hypothesis is that the points lie in $E^n$ guarantees that the problem will have at least one solution. However, whether there is a unique solution, more than one solution, or a manifold of solutions, depends on the sparsity pattern of $D^*$.

At the current state of the art of NMR, the accuracy of the measurements of the principally observable distances is limited, so that these distances are known to lie only in a certain range. We can always choose a value for $d_{ij}$ between the upper and lower bound (e.g., the central value) but, in this case, embeddability in $E^3$ is no longer guaranteed. We thus have the general incomplete embedding problem.

Given a partially filled matrix $D^*$ of interpoint distances, find a complete matrix $D$ that is embeddable in $E^n$, so that $D$ minimizes $\|D^* - D\|^2$, where $\|\cdot\|$ is a suitable norm which involves elements $d_{ij}^2$ of $D^*$ and $d_{ij}$ of $D$ only for which $(i, j) \in I$.

It should be noted that the special incomplete embedding problem is a special case of the general incomplete embedding problem for which the existence of a $D$ with $\|D^* - D\| = 0$ is guaranteed; i.e., for which the distances are accurate, and the structure is embeddable in $E^n$. It should also be noted that the calculation of cartesian coordinates from the complete distance matrix $D$—i.e., the embedding of $D$—is straightforward (2).

Since the general incomplete embedding problem is a problem of optimization, a natural approach to its solution is to minimize the object function

$$\|D^* - D\|^2 = \sum_{i,j \in I} (d_{ij}^* - d_{ij})^2. \tag{1}$$

Approaches to minimize Eq. 1 differ in how $d_{ij}^*$ is expressed—i.e., in the coordinate system in which the optimization is carried out. A popular choice is the set of cartesian coordinates (3, 4):

$$d_{ij}^* = \sum_{k=1}^{n} (x_{ik} - x_{jk})^2. \tag{2}$$

Another choice would be the torsion angles $\theta = \theta_1, \ldots, \theta_n$, which can be taken as the only internal degrees of freedom of a molecule:

$$d_{ij}^* = \sum_{k=1}^{n} |x_{ik}(\theta) - x_{jk}(\theta)|^2. \tag{3}$$

The function in Eq. 1, with $d_{ij}^*$ from Eqs. 2 or 3, generally has many local minima. The number of local minima increases with increasing sparsity of $D^*$. Since, at the present time, only local minimizers are available, minimization of Eq. 1 does not provide the optimal solution of the general incomplete embedding problem in general.

In this paper, we investigate the possibility of using distances as direct variables in the minimization of Eq. 1. Our hope is that this will yield a better convergence of the optimization of Eq. 1 toward the optimal solution.

In general, to embed a structure in a (usually) euclidian space of any desired dimension, certain relations (constraints) between the distances must exist. Distance geometry provides relations to deal with these constraints. In euclidian spaces, these relations take the form of Cayley–Menger (CM) determinants, but these are analytically complex and involve high computational cost.

In this paper, we present a simplified distance geometry for euclidian spaces. By applying Jacobi’s theorem (5, 6), we can reduce the complexity of a major type of determinant needed in the distance geometry of euclidian spaces. The theorems of distance geometry then lead to the concept of CM coordinates, in terms of which each point in $E^n$ is represented by $n + 1$ distances to $n + 1$ points of a reference structure. The CM coordinates can ultimately be used to treat the problem of finding an incomplete distance matrix so that our previous procedure (2) can be applied to embed

Abbreviations: CM, Cayley–Menger; EST, equal-sided tetrahedron; CP, cartesian pyramid.
§To whom reprint requests should be addressed.
the corresponding structure in a three-dimensional space. For this purpose, we show here how the CM coordinates can be transformed to cartesian coordinates by applying our previous embedding procedure (2).

We begin with a brief review of the relevant theorems in distance geometry.

**CM Determinants**

The proofs of Theorems 1–4 and a detailed discussion thereof have been given by Blumenthal (7).

The squared volume $V^2(p_0, \ldots, p_k)$ of a $k$-dimensional simplex, with $k + 1$ points $p_0, p_1, \ldots, p_k$ can be expressed entirely in terms of distances:

$$V^2(p_0, \ldots, p_k) = \left( \frac{(-1)^{k+1}}{2^k (k!)} \right)^2 \begin{vmatrix} 0 & 1 & 1 & \ldots & 1 \\ 1 & s_{01} & s_{02} & \ldots & s_{0k} \\ 1 & s_{10} & 0 & \ldots & s_{1k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & s_{k0} & s_{k1} & \ldots & 0 \end{vmatrix}, \quad [4]$$

where $s_{ij} = d_{ij}^2$ and $d_{ij}$ is the distance between points $i$ and $j$ of the simplex. The determinant on the right-hand side of Eq. 4, which is the matrix of squared distances of the simplex bordered by a row and column of 1s with intersecting element 0, is called the CM determinant of the simplex $(p_0, p_1, \ldots, p_k)$.

For euclidian spaces, the squared volume has to be positive and we therefore have the following.

**THEOREM 1.** A necessary and sufficient condition that the distance matrix

$$D(p_0, \ldots, p_n) = \begin{pmatrix} 0 & d_{01} & \ldots & d_{0n} \\ d_{10} & 0 & \ldots & d_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{n0} & d_{n1} & \ldots & 0 \end{pmatrix}$$

of $n + 1$ points $p_0, \ldots, p_n$ is embeddable in a euclidian space $E^n$ of dimension $n$ is that, for every $k = 1, 2, \ldots, n$, the sign of $CM(p_0, \ldots, p_k)$ is $(-1)^{k+1}$, i.e., $(-1)^{k+1} CM(p_0, \ldots, p_k) \geq 0$.

If $n + 2$ points are embeddable in $E^n$, then their $(n + 1)$-dimensional volume must vanish, and we have the following.

**THEOREM 2.** If $D(p_0, \ldots, p_n)$ is embeddable in $E^n$ (i.e., if Theorem 1 holds), then a necessary and sufficient condition that $D(p_0, \ldots, p_n; p_{n+1})$ is embeddable in $E^n$ is that $CM(p_0, \ldots, p_n; p_{n+1}) = 0$.

Similarly, in $E^n$ the $(n + 1)$- and $(n + 2)$-dimensional volumes of $n + 3$ points have to vanish.

**THEOREM 3.** If $D(p_0, \ldots, p_n)$ is embeddable in $E^n$ (Theorem 1), then a necessary and sufficient condition that $D(p_0, \ldots, p_n; p_{n+1}; p_{n+2})$ is embeddable in $E^n$ is that

$$CM(p_0, \ldots, p_n; p_{n+1}; p_{n+2}) = 0.$$

Combining Theorems 1–3, we obtain the global embedding theorem, which governs the embeddability of a distance matrix $D(p_0, \ldots, p_m)$ in a euclidian space $E^n$.

**THEOREM 4** (global theorem). A distance matrix of $h + 1$ points $D(p_0, p_1, \ldots, p_h)$ is embeddable in a euclidian space $E^n$ of dimension $n$ if and only if

(i) There exists a submatrix $R(p_0, p_1, \ldots, p_n)$ of $n + 1$ points of $D$ so that, for every $k = 1, 2, \ldots, n$ the sign of $CM(p_0, \ldots, p_k)$ is equal to $(-1)^{k+1}$;

(ii) For every pair $p_i, p_j$ (i.e., $n + 1$, \ldots, $h$)

$$CM(p_0, \ldots, p_n; p_i) = 0, \quad CM(p_0, \ldots, p_n; p_j) = 0, \quad CM(p_0, \ldots, p_n; p_i; p_j) = 0.$$

**CM Coordinates**

Theorem 4 splits the simplex $(p_0, \ldots, p_m)$, where $(m + n) = h$, into two pieces. The $n + 1$ points $(r_0, \ldots, r_n)$ of part $i$ of Theorem 4 are used as a reference for the points $(p_{n+1}, \ldots, p_{m+1})$, which we shall call the object. As illustrated in Fig. 1, the matrix of squared distances is partitioned into four pieces: the squared distances $r_{ij}$ within the reference, the squared distances $s_{ij}$ within the object, and the squared distances $t_{ij}$ between the object and the reference.

The squared distances $t_{ij}$ can be regarded as coordinates for point $p_i$ of the object with respect to the reference $R = (r_0, \ldots, r_n)$. We shall call the squared distances, $t_{ij}$, the CM coordinates for point $p_j$ with respect to the reference $R$. Since the reference in $E^3$ is a triangle and the reference in $E^3$ is a tetrahedron, we shall use the terms triangular and tetrahedral coordinates, respectively, in these cases (Fig. 2).

So far we have considered the reference as being part of the structure (distance matrix), but we can also add an external reference to the object. From this point of view, the CM coordinates describe the orientation of the object with respect to the reference, similar to a cartesian frame, where a geometric object is referred to a set of orthogonal axes together with an origin.

The CM coordinates are not independent. For any point $p_i$ of the object, the $n + 1$ CM coordinates satisfy the relation

$$CM(R; p_i) = 0. \quad [5]$$

On the other hand, the squared distances $s_{ij}$ of the object are functions of the CM coordinates $p_i$ and $p_j$ through the condition

$$CM(R; p_i, p_j) = 0. \quad [6]$$

This relationship was also used by Klapper and DeBrotka (8). In $E^3$, these two types of (five- and six-point) determinants are

$$CM(R; p_i) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12} & r_{13} & r_{14} \\ 1 & r_{21} & 0 & r_{23} & r_{24} \\ 1 & r_{31} & r_{32} & 0 & r_{34} \\ 1 & r_{41} & r_{42} & r_{43} & 0 \end{vmatrix}$$

and

$$CM(R; p_i, p_j) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12} & r_{13} & r_{14} & t_{1j} \\ 1 & r_{21} & 0 & r_{23} & r_{24} & t_{2j} \\ 1 & r_{31} & r_{32} & 0 & r_{34} & t_{3j} \\ 1 & r_{41} & r_{42} & r_{43} & 0 & t_{4j} \\ 1 & t_{1j} & t_{2j} & t_{3j} & t_{4j} & 0 \end{vmatrix}. \quad [7]$$

These determinants must vanish if the six points are to be embeddable in $E^3$. For an object of $m$ points, there are $m$
detemrants $CM(R; p_j)$ and $m(m - 1)/2$ determinants
$CM(R; p_i, p_j)$.

Analytical Form of the CM Determinants

To illustrate the computational complexity to evaluate the
CM determinants $CM(R; p_j)$ and $CM(R; p_i, p_j)$, we choose a simple type of reference, an equal-sided tetrahedron with
eight length $r_{a,b} = 1(\alpha \neq \beta, \alpha, \beta = 1, ..., 4)$ in $E^3$. Expansion
of $CM(R; p_j)$ (Eq. 7) yields

$$CM(R; p_j) = 4\left(\sum_{i=1}^{4} t_{ij} + 1\right) - \left(\sum_{k=1}^{4} t_{kk} + 1\right)^2.$$  [9]

The analogous expression for $CM(R; p_i, p_j)$ (Eq. 8) is too
lengthy to be reported here and, therefore, the resulting quadratic
equation for $s_{ij}$ is also complicated. However, we can simplify $CM(R; p_i, p_j)$ and, in fact, replace it by a simpler
expression using the help of the following theorem.

**Theorem 5** [Jacobi’s theorem (5, 6)]. Let $A$ be a square
matrix with $p$ rows and columns and $A^{ad}$ be its adjugate
matrix, whose elements $a_{ij}$ are the algebraic complements
of the elements $s_{ij}$ of $A$. Let $A$ be decomposed in blocks as shown in Fig. 3, and similarly for $A^{ad}$.

$$|A|^{p-s-1} = \frac{(\alpha_{n+3,n+3} \alpha_{n+4,n+4} - \alpha_{n+3,n+4} \alpha_{n+4,n+3})}{|R|}.$$  [12]

Since $p = n + 4$ and $s = n + 2$, the exponent on the left-hand
side is unity. The determinant $|R|$ is $CM(R)$, the CM
determinant of the reference, and therefore, except for the numerical
factor of Eq. 4, is the squared volume of the reference. For $CM(R; p_i, p_j)$ to be finite $CM(R)$ must not vanish.

The elements $\alpha_{n+3,n+3}$ and $\alpha_{n+4,n+4}$ are $CM(R; p_j)$ and
$CM(R; p_i)$, respectively. Since $CM(R; p_i, p_j)$ is a symmetric
determinant, we have $\alpha_{n+3,n+4} = \alpha_{n+4,n+3} = CM^*(R; p_i, p_j)$ and

$$CM^*(R; p_i, p_j) = \left|\begin{array}{cccccccc}
0 & 1 & 1 & ... & 1 & 1 \\
1 & r_{12} & ... & r_{1,n+1} & t_{1i} & t_{1j} \\
1 & r_{21} & ... & r_{2,n+1} & t_{2i} & t_{2j} \\
... & ... & ... & ... & ... & ... \\
1 & t_{n+1,1} & ... & 0 & t_{n+1,i} & t_{n+1,j} \\
1 & t_{ij} & t_{ij} & ... & t_{n+1,i} & s_{ij} & 0
\end{array}\right|.$$  [13]

Substituting the appropriate determinants in Eq. 12, we obtain

**Fig. 2.** CM coordinates in $E^3$ (triangular coordinates) for the
object (two points $p_i$ and $p_j$) with respect to the triangle [reference; ($r_1$, $r_2$, $r_3$)].

**Fig. 3.** Decomposition of the symmetric matrix $A$ and its adja-
cute matrix $A^{ad}$ into blocks for Jacobi’s theorem. The theorem
makes use only of $A$ and the shaded blocks $R$ and $Y$. (Adapted from ref. 6.)
Theorem 4' (global embedding theorem). The necessary and sufficient conditions that an \((m + n + 1) \times (m + n + 1)\) matrix \(D\) of distances \(d_{ij}\) (i, j = 0, 1, ..., m) is embeddable in a euclidean space \(E^n\) of dimension \(n\) are as follows:

(i) There exists an \((n + 1) \times (n + 1)\) submatrix \(R\) of \(D\) corresponding to \(n + 1\) points \(\{r_1, r_2, ..., r_{n+1}\}\) — i.e., a reference — that is embeddable in \(E^n\) (Theorem 1) and whose \(n\)-dimensional volume does not vanish.

(ii) All determinants \(CM^*(R; p_i, p_j)\), \(i, j = 1, 2, ..., m\), \(i \geq j\), vanish. If Theorem 4' holds, then (according to Eq. 18) the squared distances \(s_{ij}\) in the object \((p_1, ..., p_m)\) are functions of the CM coordinates of \(p_i\) and \(p_j\)

\[
s_{ij} = -[t_i^T A t_j + b_i^T(t_i + t_j) + C ]/CM(R). \quad [20]
\]

We note that the CM variables are squared distances. Since every distance in \(E^n\) between the reference and the object has to be real and since the only constraint on the CM coordinates for point \(i\) (the squared distances between \(p_i\) and the points of the reference) is that \(CM^*(R; p_i, p_j)\) shall vanish, we have the following.

Corollary 1. If \(R\) is a reference in \(E^n\) and if \(CM^*(R; p_i, p_j) = 0\), then \(t_i = 0\) \((k = 1, 2, ..., n + 1)\); i.e., for every \(n\)-dimensional euclidean reference \(R\), the vanishing of \(CM^*(R; p_i, p_j)\) ensures that the distances \((t_{ik})^{1/2}\) from the reference to \(p_i\) are real.

Also, since the squared distances \(s_{ij}\) of the object are functions of the CM coordinates of \(p_i\) and \(p_j\), we have the following.

Corollary 2. If \(R\) is an \(n\)-dimensional euclidean reference and if \(CM^*(R; p_i, p_j) = 0\), then

\[
d_{ij}^2 = s_{ij} = -(t_i^T A t_j + b_i^T(t_i + t_j) + C )/CM(R) \geq 0;
\]

i.e., \((s_{ij})^{1/2} = d_{ij}\) is real.

Transformation of CM Coordinates to Cartesian Coordinates

In a recent paper (2), we showed that any matrix \(D^2\) of squared distances \(d_{ij}^2\) with \(d_{ii}^2 = d_{jj}^2 = 0\) \((i, j = 1, ..., l + 1)\) is embeddable in a pseudo-euclidean space \(\mathbb{P}^q\) \((p + q = l)\) with \(p\) real and \(q\) imaginary dimensions, and we reported an embedding procedure to calculate cartesian coordinates from \(D^2\).

Since the squared distances \(r_{\alpha \beta}\) of the reference together with the CM coordinates \(t_{\alpha \beta}\) and the squared distances \(s_{ij}\) of the object form such a matrix, we can apply the embedding procedure to calculate cartesian coordinates from the CM coordinates.

Since the conditions of Theorem 4' guarantee that

\[
D^2 = \begin{pmatrix} R & T \\ T^T & S \end{pmatrix}
\]

will be embeddable in \(E^n\), the embedding procedure will stop after \(n\) projections, yielding \(n\) coordinates for the \(n + 1\) points of the reference and the \(m\) points of the object. Also, since \(R\) is embeddable in \(E^n\), we have \(r_{\alpha \mu + 1} > 0\) \((\mu = 1, ..., n)\), \(n\) where \(\mu\) is the projection number, so that we can use this specific sequence of generators for the \(n\) projections. It should be noted that the squared distances \(s_{ij}\) of the object will not enter the embedding procedure, since embedding will be completed after \(n\) projections, so that only the blocks \(R\) and \(T\) (but not \(S\)) of \(D^2\) are needed for embedding.

Examples

We calculate the analytical forms Eq. 18 for two references in \(E^3\), the equal-sided tetrahedron EST with unit edge \(r_{\alpha \beta} = \ldots\)
1 (α ≠ β) and the cartesian pyramid CP, which has the following matrix of squared interpoint distances

\[
CP = \begin{pmatrix}
 0 & 1 & 1 & 1 \\
 1 & 0 & 2 & 2 \\
 1 & 2 & 0 & 2 \\
 1 & 2 & 2 & 0
\end{pmatrix}.
\]  \[22\]

CP mimics a cartesian reference frame, since point r1 can be considered as the origin and the edges r12, r13, and r14, as the axes of a cartesian frame.

For EST, A, b, C, and CM(R) of CM*(EST, p1, p3) of Eq. 18 are

\[
A = \begin{pmatrix}
 3 & -1 & -1 & -1 \\
 -1 & 3 & -1 & -1 \\
 -1 & -1 & 3 & -1 \\
 -1 & -1 & -1 & 3
\end{pmatrix}, \quad b = \begin{pmatrix}
 -1 \\
 -1 \\
 -1 \\
 -1
\end{pmatrix}, \quad \text{[23]}
\]

\[C = 3, \text{ and } CM(R) = 4. \text{ Substitution of these constant parts into Eq. 18, and expansion, yields Eq. 16.} \]

Having obtained the CM coordinates, we now resort to the embedding procedure of our earlier paper (2) to obtain the cartesian coordinates as a function of the CM coordinates. For EST, we obtain

\[
x_i = \frac{1}{2} (t_{1i} - t_{2i} + 1)
\]

\[
y_i = \frac{1}{2(3^{1/2})} (t_{1i} + t_{2i} - 2t_{3i} + 1)
\]

\[
z_i = \frac{1}{2(6^{1/2})} (t_{1i} + t_{2i} + t_{3i} - 3t_{4i} + 1).
\]  \[24\]

For CP, we obtain

\[
A = \begin{pmatrix}
 12 & -4 & -4 & -4 \\
 -4 & 0 & 0 & 0 \\
 -4 & 0 & 0 & 4 \\
 -4 & 0 & 0 & 4
\end{pmatrix}, \quad b = \begin{pmatrix}
 +4 \\
 -4 \\
 -4 \\
 -4
\end{pmatrix}, \quad \text{[25]}
\]

\[C = 12, \text{ and } CM(CP) = 8, \text{ and the cartesian coordinates in terms of the CM coordinates are obtained as} \]

\[
x_i = \frac{1}{2} (t_{1i} - t_{2i} + 1)
\]

\[
y_i = \frac{1}{2} (t_{1i} - t_{3i} + 1)
\]

\[
z_i = \frac{1}{2} (t_{1i} - t_{4i} + 1). \]  \[26\]

It should be noted that, in both examples, the cartesian coordinates are linear functions of the CM coordinates, although the CM coordinates are squared distances.

**Possible Use of This Treatment for the Incomplete Embedding Problem**

We shall now show that the CM determinants CM*(R; p1, p3) of Theorem 4′ yield a suitable norm for the general incomplete embedding problem. The idea is to add an external n-dimensional reference to the incomplete matrix D* and to establish CM coordinates for the points p1 of D* that approximate the d′̅ by means of Eq. 18; i.e., we want the s̅∥s, calculated from the CM coordinates, to come as close as possible to the s̅∥s, as shown below.

For an arbitrary set of values t̅∥(k = 1, ..., n + 1, i = 1, ..., m), the conditions for embeddability in part ii of Theorem 4 will not be satisfied in general. But we can define a matrix D2 with elements s∥∥ by means of Eq. 20 for any set of values of CM coordinates t̅∥. D will be embeddable in E2 if and only if CM(R; p1, p3) = 0 (i = 1, ..., m) since only then does Eq. 20 yield s∥ = 0.

For s∥ = (d∥)^2, Eq. 18 yields

\[
CM*(R; p1, p3) = t^T A t + b^T(t + t̅) + C + CM(R)s∥. \]  \[27\]

From Eq. 18, we have

\[-CM(R)s∥ = t^T A t + b^T(t + t̅) + C, \]  \[28\]

which, when inserted in Eq. 27, yields

\[CM*(R; p1, p3) = CM(R)(s∥ - s∥). \]  \[29\]

Since s∥ = 0, we have for i = j

\[CM*(R; p1, p3) = CM(R)s∥. \]  \[30\]

Eq. 29 therefore leads to the norm

\[
||D* - D||^2 = \sum_{i,j \in I} [CM*(R; p1, p3)]^2 = CM(R)^2 \sum_{i,j \in I} (s∥ - s∥)^2. \]  \[31\]

It should be noted that the pairs (i, i) are always in I, since d̅ = 0. The best solution of the general incomplete embedding problem for the class of all matrices with elements d∥ = (d∥)^2 defined by Eq. 28 is therefore the minimum of ||D* - D||^2 in Eq. 31.

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